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WS-8 Concepts and their names

When you evoke a concept by its name, you make out of this name a signifier that represents you as a subject to another signifier which is inserted in the language code. When you do this, we say that you have *learned the concep*t and is contributing to the production of its meaning. The *concep*t is the positing of meaning. The set of meanings of a concept at a given time we call *conception*. The concept dealt with here, is *number.*

In your speeches we have been able to discern two conceptions of the number line. They are both legitimate conceptions of 20th century mathematics (M20): the *standard conception*, which only recognizes real numbers ~ , and the *nonstandard conception*, which includes hyper real numbers *~ beyond ~ . Among the hyper real numbers (the hyper reals) there are infinitesimal and infinite numbers.

The standard conception

~ is the real line with the usual operations of addition and multiplication, together with the order relation.

$\Delta F = F(t+\Delta t) - F(t)$ is the *variation of the function*; Δt is a real number called *increment*; it may be positive or negative. The variation of the function is *final value minus initial value*.

$$\frac{\Delta F}{\Delta t} = \frac{F(t+\Delta t) - F(t)}{\Delta t}$$

is the *variation rate* of the function, also called the *mean variation rate* or still, *Newton's quotient*.

$\lim\limits_{\Delta t \to 0} \frac{\Delta F}{\Delta t} = \lim\limits_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t}$ is called the *derivative of F*. It is written

$$\boxed{\frac{dF}{dt} = F'(t)}$$

The *accumulated area* under the graph of a function *f*, between *a* and *b*, will be written Af_a^b It is also called the *definite integral* of *f* and written $Af_a^b = \int_a^b f(t)dt$.

"Accumulated" means that the areas under the t-axis must be subtracted, that is, counted as negative.

Important: In the standard conception, $\frac{dF}{dt}$ e $\int\limits_a^b f(t)dt$ are symbols: neither the first is a quotient nor the second contains a product or is a sum.

In the standard conception, many students puzzle with the question: does the limit finally arrives or does not arrive? To remain in the standard conception, we must ask the student: *arrives where?* In this conception, the limit is the target of the process of "tending to". *The limit points to the place where a presence is inscribed*. Forget the place and keep the presence. For instance, 0.999… points to the place where 1 is inscribed; 0,999… *is* one.

Saying that "the limit tends to" denotes poor understanding. *The limit is*. For instance,

$$0,999... := \lim_{n\to\infty} 0,\underbrace{999...9}_{n} \text{ is (equal to) one}$$

If you want to refer to the process of *adding one more* 9, you may say that 0,999… *tends to one*. In this case, you should write

$$0,\underbrace{999...9}_{n} \underset{n\to\infty}{\rightarrow} 1$$

Then, when then student asks whether the limit arrives or does not arrive, we pass to the nonstandard conception by answering: *before it arrives, it becomes infinitely close*.

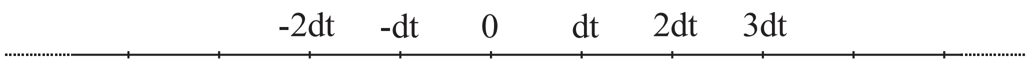
The non standard conception

This conception is originally due to Leibniz who invented the notation still used today, in around 1700. The standard conception was developed by Cauchy and later by Weierstrass from 1824 to 1880. Mathematicians struggled to avoid references to geometry and to infinitesimals. These generated many mistakes in he ranks of second-class scientists. However, infinitesimals survived in applied sciences. In mathematics they survived as a clandestine concept. The meaning of infinitesimals last in commonsense; many students express ideas about them, such as this one: *an infinitesimal is zero point zero zero, infinitely many zeros and then one*.

In 1960, Abraham Robinson showed that infinitesimals can be expressed within the standard conception. Thereby he opened a branch of mathematics called *nonstandard analysis*. However, whatever can be proved with infinitesimals, can also be proved via standard arguments. The advantage of nonstandard analysis lies in teaching and condensing speeches.

Na concepção não standard, a reta numérica fica "mais grossa": ela passa a incluir, além dos números reais, números infinitamente próximos a eles e números infinitos, chamados todos números hiper-reais formando a reta hiper-real anotada *~ . Each real number is surrounded by a cloud of hyper-reals infinitely close to it, called its *monad.*

If *t* is the variable of a hyper-real axis *~ , *dt* denotes an nfinitesimal of *t*, positive or negative. Is absolute value is smaller than any positive real number. If *t* means time, *dt* is an infinitesimal duration; if *t* means distance, *dt* is an infinitesimal displacement. In any case, *dt* is an *increment*. The 4000 times zoom of CorelDraw gives an idea of an infinitesimal world, which is of another order of magnitude than our finite world. However, the CorelDraw only reduces 4000 times, producing an illusion of infinitesimals. What we have seen there, are not infinitesimals, but only very small line segments as compared to the size of the page. If we could dive into the infinitesimal world of *dt* we would see something like this:



Here, *dt* would work as 1 of na new hyper-real line, with infinitesimals of a higher order, like $(dt)^2$ which is infinitely smaller than *dt*. Besides, there would also be infinite numbers in the world of *dt*, like $\frac{dt}{dt}$, which would be the 1 of our finite world and $\sqrt{dt}$ which would be infinite in our world. Actually, *~ *reproduces itself as a fractal*. The set of all infinitesimals constitute the monad of zero.

Transporting the monad of zero to a real number we get the monad of this real number. therefore, in *~ each real number is the center of a monad consisting of hyper-reals infinitely close to it and, consequently, infinitely close to each other. The sign $\approx$ stands for infinitely close. For instance, $dt \approx 0$, $1 + dt \approx 1$, etc. Different real numbers have disjoint monads.

In the nonstandard conception, the above notation acquires new meanings. In $\Delta F = F(t+\Delta t) - F(t)$ we replace Δt by *dt* and get the basic definition of *differential* of a function[1]:

$$\boxed{dF = F(t+dt) - F(t)}$$

We say that *dF* is the differential of *F* and *dt* is the differential of *t*. If necessary, we may write $dF = dF(t, dt)$ to emphasize that *dF* depends on *t* and *dt*. Newton's quotient is now the quotient of two infinitesimals.

$$\frac{dF}{dt} = \frac{F(t+dt) - F(t)}{dt}$$

This quotient is a hyper-real number. If it is finite, it must be in the monad of some real number. This real number is called the *derivative of F at t* and is written $F'(t)$. [2] Therefore:

$$\frac{dF}{dt} = \frac{F(t+dt) - F(t)}{dt} \approx F'(t)$$

The above symbol $\frac{dF}{dt} = F'(t)$ is now a formula, $\boxed{\frac{dF}{dt} \approx F'(t)}$ that is, the dash is a true quotient between infinitesimals. Consequently, we may write $\boxed{dF \approx F'(t)dt}$ as a true product. Denoting $F'(t) = f(t)$, the nonstandard conception allows us to write:

$$dF \approx F'(t)dt = f(t)dt$$

[1] If an infinitesimal variation of $t \in$ ~ produces na infinitesimal variation of *F*, we follow Cauchy and say that *F* is continuous at *t*.
[2] When his happens, we say that the function is differentiable at *t*.

It becomes evident that *dF*, the infinitesimal variation of *F*, is the area of a rectangle with infinitesimal basis *dt* and height $f(t)$ under the graph of f. Furthermore,

$\frac{dF}{dt} \approx f(t)$ expresses that the derivative of the running total is the daily deaths, thereby formalizing the result of WS-9.

Writing $\langle a_n \rangle$ for the hyper-real number determined by the sequence $\{a_n\}$, we may say that:

$$\left\langle 0,\underbrace{999...9}_{\text{n noves}} \right\rangle \overset{\approx}{<} \left\langle 0,\underbrace{99\,99\,99...99}_{\text{n pares de noves}} \right\rangle \overset{\approx}{<} 1$$