GAMES FOR INTEGERS: CONCEPTUAL OR SEMANTIC FIELDS?[1]

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ABSTRACT

By the ***sign rule problem*** we understand four questions into which we cast Glaeser's historical survey [Glaeser, 1981] and Brousseau's epistemological remarks [Brousseau 1983] about integers: *How to take the bigger from the smaller? How to subtract a negative? Why minus times minus equals plus? What does it mean minus-times something?* In the paper we present a didactical strategy to solve this problem, based on Baudrillard's conception of *game* [Baudrillard, 1979] and on the theory of Conceptual Fields (**CF**) [Vergnaud, 1990]. We report some experimental results and discuss them from the point of view of the theory of Semantic Fields (**SF**) [Lins, 1994].

Difficulties about integers are quite old. In his historical survey, Glaeser [1981] describes perplexities of famous mathematicians of the past about the sign rule. The proof that we know today was first given by Haenkel in 1867 in a text about complex variables. We know that it is useless as an explanation for convincing a 13-years old student. Integers have scarcely been dealt with in recent literature. Among 56 research reports presented in PME XVIII, only one explicitly concerns integers [Lytle, 1994]. The sign rule remains a major problem for the teacher.

Works about integers generally display a profusion of suggestions for addition but are insufficient about multiplication. Glaeser [1981] points out this insufficiency in Freudenthal [1973]. "The reading of pages 279/281 does not even suggest that he has realized the astonishing phenomenon studied here" [p. 305]. Freudenthal [1983] offers three simultaneous approaches to the sign rule problem. The first insists on the necessity of permanence of distributive and commutative laws $(-3)\times4=4\times(-3)$ [p. 434]. This leads to the usual difficulties: students keep asking: *what does it **mean minus-three times** something? Less than zero times it?* The second approach is extension of linear transformations according to what he names the "geometrical-algebraical permanence principle" [ib. id., p. 444]. This raises the problematic relation between discrete-numerical and continuous-geometrical domains. The third approach is simply teaching rules, among which (-a).(-b)=a.b. Freudenthal asks for "the most simple and effective way to programme the learner with (...) six rules. It is almost nothing compared with the rules a child must learn in order do master

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a column arithmetic" [sic. p. 457]. When the student comes to the point of asking why minus times minus equals plus it is already too late[4]: he has learned a solution without knowing the problem and is "fed up" with rules.

We thought of anticipating the solution to the sign rule problem as *theorems in action*, according to CF. Our idea was that the roles should be exchanged: the teacher should be the one to ask and the student the one to answer why *minus times minus makes plus*. The didactical strategy should lead the student to provide his own explanation to facts that he should consider as evident: "(theorems in action) are associate with a feeling of obviousness: they are (...) taken as obvious properties of situations" [Vergnaud, 1982, p. 36].

This research was aimed at testing the didactical validity of a certain conception of **game** to solve the sign rule problem. This conception sharply distinguishes *game* from *activity*. At the moment of the game, nothing else counts but following a rule. "The game is the vertigo of the rule. By choosing a rule we suspend the law. The obligation that the game creates is of the order of a challenge" [Baudrillard, 1979, p. 151]. In particular, no interruptions for registering results or making connections with the syllabus should be admitted. In designing *games for integers* we have been guided by pedagogical beliefs that are best stated as answers to two questions. Question 1: "How can we make theorems become theorems in action?" [Vergnaud, 1982, p. 36]. Our answer was: By engaging the student in *games* where the use of theorems in action leads to better playing strategies. Question 2: "How can we make theorems in action become theorems?" [ib. id.] Our answer was: By introducing adequate work-sheet activities based on the game, after it has been finished.

THE DIDACTICAL PROBLEMS AND THEIR SOLUTIONS

Glaeser's [1981] historical account of epistemological obstacles in the development of integers was made more precise by Brousseau [1983]. We cast these works into four didactical problems:

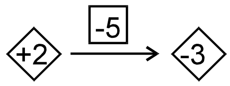
P1. How to take the bigger from the smaller? 3-5=...
P2. How to subtract a negative? -(-3)=...
P3. Why minus times minus equals plus? (-2)(-3)=...
P4. What does it mean *minus ... times something*? (-3)× ...?

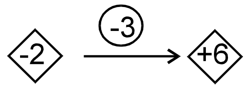
What do we mean by solutions to these problems? Our premise has been that i**ntegers are operators on signed naturals** and operations between integers are operations with such operators.

□ ○

[4] This statement should be taken as rethorical but we believe that it can be justified.

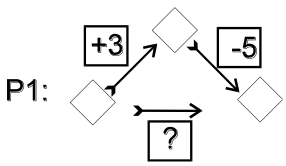
In a state-operator conception, using for additive, for multiplicative operators and for their states, our two basic diagrams are:

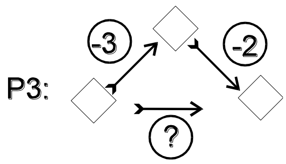


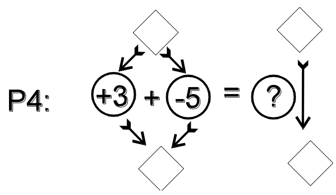


We emphasize that the ± signs before the operator numbers do not have the same meaning as the ± igns before the state numbers. This is easily seen by noting that we can use a red/blue code for the operators and a ± code for their states or vice versa. Denoting the multiplicative operators by f_a , the second diagram represents the transformation of one state into another: f_{-3} (-2) = +6. “To multiply a rational integer by -*a* is to multiply by *a* and change the sign of the product” [Papy, 1968, p. 334]. We argue that this is not yet the solution of *minus times minus*. P3 is solved when the subjects perform the composition of multiplicative operators: $f_{-3} \circ f_{-2} = f_{+6}$ which is quite different. In our games we have introduced f_{-3} (-2) = +6 as a rule and expected that $f_{-3} \circ f_{-2} = f_{+6}$ would come out as a theorem in action. Note that in both cases the calculation (-3)×(-2) = +6 is the same.

By solutions of the four didactical problems (theorems) we mean the following four theorems in action. By a solution of P2 we mean the action of removing a debt by increasing the amount of money in a gain/debts model. By solutions of P1, P3 and P4 we mean the actions of completing the diagrams below without resorting to states.

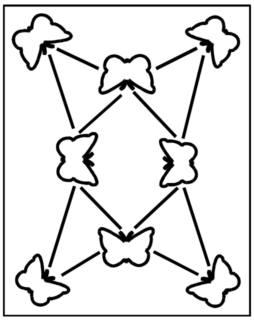




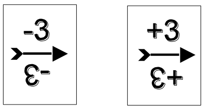


THE DIDACTICAL STRATEGY: THE GAMES

According to Brousseau’s conception of *learning* we cannot hope to solve these problems in one stroke: “learning is the result of experimentation of successive conceptions, temporarily and relatively good that have to be successively rejected or retaken in a truly new genesis at each time” [Brousseau, 1983, p. 171]. In order to provide conditions for experimentation of conceptions, we designed three games based on additive and multiplicative machines [Dienes, 1969A, 1969B].

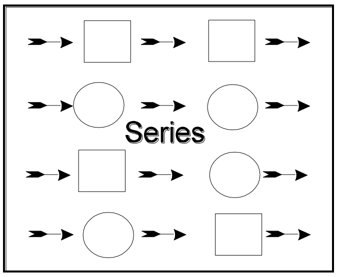


G1. This is an additive state-operator game. The kit consists of a board with a stamped network, cards representing additive operators to be placed on the network's branches, and one-color beads, representing the operator states, to be placed on the network's knots (butterflies). Two signs are associated with card numbers: an operative sign (an arrow) and a predicative sign (+/-). Operator states <u>are</u> <u>not</u> signed. The objective of the game is to place the cards between knots so as to close commutative circuits. In the <u>advanced</u> <u>version</u>, players should develop schemes for composing additive operators <u>without</u> resorting to the beads. The expected strategy is direct composition of

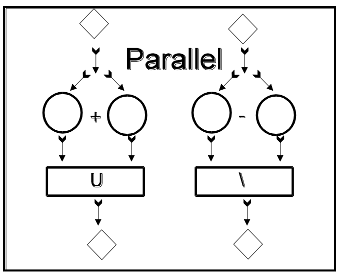


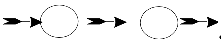
operators, thus solving P1. We denote this strategy by **G1A**. In applying this game we have observed the emergence of a strategy consisting in mentally keeping track of the states by memorizing the number of beads in one fixed knot, generally the first one that was filled. We denote this strategy by **G1B**. A puzzle occurs when a card is to be placed in such a way as to close two circuits simultaneously and, due to a previous unnoticed mistake, the numbers assigned to it from each of the circuits do not match. We shall refer to this situation as **G1C**. In some cases the number of beads on a knot is not sufficient to allow for the subtraction determined by the card that a player wants to place. It is expected that they decide to increase the number of beads on all previously filled knots. We refer to this solution as **G1D**.

G2. This is a real estate sales game with "red money" representing debts. This game introduces <u>signed</u> <u>operator</u> <u>states</u>. Instruction cards may ask a player to "*remove a $10 red bill*" from the stock of a partner who happens to have no debts. This situation installs the *neutralization of opposites* that solves P2 [Lytle, 1994]. We call it **G2A**.



G3. This is a pawn-track game. The position number to which the pawn has to be moved is obtained from the introduction of its present position number into the entry of a connection of additive and multiplicative *machines.* The track numbers are coded +/-. The operator numbers to be placed in the machines are obtained by throwing two dice with face numbers also coded +/-. To negative multiplicative



operators is assigned the property of reversing the sign of the states upon which they act. The player may choose the machine connection best suited for his move. The objective is to put pawns on the positions numbered, say, 30. In the first match only series connections may be used; in the second one, only parallel connections. We define **G3A** as the strategy consisting in operating with the dices' numbers before trying the series connection of multiplicative operators . This is the solution to P3. Moving his pawn to a position whose number does not divide 30 indicates that the player has not solved P4. When the player avoids this trap we say that he has developed **G3B**.

THE SITUATIONS

Informal experiments and adjustments were made so as to fit the kits to the *vertigo of the rule* conception of game. Then a systematic teaching experience was made with G1 and observations were made with G1 and G2.

S1. Two of us applied the G1 and subsequent work-sheets in two out of six classes a week for 5th graders (11 years-old) during 7 weeks in a public school of a relatively rich town of São Paulo, Brazil. Officially the study of integers should only begin at 12 but the school administration raised no objections. The class was being taught by one of the authors. Each meeting lasted 1.5 hour. The class was divided into 9 groups of 4 students. Each group received a complete G1 kit. First, beads were used on the knots (butterflies). Then we asked the children to leave them out (advanced game version). Special care was taken not to hint how to perform direct composition of additive operators when beads were left out. Promotional grades were set on criteria of *presence*, *participation* and *performance*. A contest was organized, awarding medals to the three first places but not being graded. The last three meetings were dedicated to group-work on four work-sheets. These activities started with problems of completing a circuit that reproduced part of the network board and ended with problems of replacing a series of cards by a single one in the absence of any butterfly drawings. We asked for solutions of the problems but suggestions were limited to: *do as you did in the game*. **S2.** One of us played G1 in two occasions, with 7 groups of mathematics high school teachers (**S2A**) and with 9 groups of college teachers and senior college students (**S2B**)[5]. The presentation was the following. If the G1A strategy did not occur we would ask the groups for a *direct method* without insisting (S2A) or increasingly insisting (S2B). If G1C occurred, we would say: *go back to the beads.* As for G1D, we would tell the solution if necessary in order to concentrate on G1A. S2A also played G3. The accorded presentation was to first play the series version and worn the players before starting the parallel version: *there is a pitfall in this game*. As they fell into the trap we would remark: *your are already trapped. Find out why*.

[5] UNIJUÍ, Ijuí (RS), November 11; Un.G., Guarulhos (SP), November 24, 1994.

THE RESULTS

R1. Two of the four contest finalists were students considered "weak" according to teachers' general opinion. The nastiest boy spontaneously worked as tutor of other groups. Only one case of cheating was observed. G1D caused no major problems; children promptly suggested the increase of the number of beads. They realized that there was some arbitrariness in this number: *you could have started (the game) with more (beads).* Nevertheless they did not develop G1A, <u>not even the winners</u>. G1B was used all the time, including on the work-sheet situation. Consequently G1C never happened. Some students resorted to writing numbers for the states on the work-sheets and then quickly erasing them while asking: *can't we play any more?* When they faced the no-butterflies problems they did not know how to proceed. At this point our suggestion: *do as in the game*, became apparently meaningless. One group that had had poor performance in the game, started developing *ad hoc* strategies, which were only successful when the cards had the same sign. The others kept asking for help. Nevertheless, considerable ability of mental calculation was developed.

R2. None of the S2B groups and only one of the S2A developed G1A, precisely the one where a member was fully aware of Dienes's works. About half of the groups ran into G1C and one third required explicit solutions for G1D. All groups faced with the demand for a direct scheme of composition of additive operators, started by reinforcing G1B. Faced with increasingly stronger demands for direct methods, four of the S2B groups succeeded, three succeeded after some feed-back information, two gave up and left the room. Three of the groups which later succeeded, chose a minimal-value state as reference for G1B. All S2A groups playing G3 fell into the G3B pitfall. Pointing out: *you are already trapped*, apparently produced little or no effects: players continued trying to reach 30 from positions such as 12. Only after repeated being told that their efforts would be useless did they produce an explanation and, in some cases, reset the game. One player praised G3 as excellent but actually rejected it as a teaching instrument: *I am going to play it with my best students outside the classroom and I will introduce some random instruction cards, so that they can get out of the trap.* Our data about G1A are not conclusive.

DISCUSSION

Results in R1 and R2 did not confirm our first pedagogical belief that theorems could become theorems in action by engaging the student in games, conceived as the *vertigo of the rule*; hence the second belief could not be checked. As foreseen by Vergnaud [1990, p. 152] and Dienes [1969A, p. 9], G1A turned out to be much more difficult than G1B. Why? There is no doubt that the different degrees of difficulty can be explained in CF in terms of complexity of the mathematical structures involved in these two problems. But we still have to explain the different outcomes of the two situations S2A and S2B, dealing with the same (G1A) problem. Mathematics teachers and senior students certainly knew P1-P4 as theorems, but they only put P1 into action in S2B and not in S2A. Nevertheless, the outcome of S2B indicates that they had the means of having done so.

The issue is not why G1A is more difficult than G1B. What we have to find out is why the difficulty of G1A was overcome in S2B and not in S2A. This is a central issue for teaching: how to overcome the difficulty imposed by a given *complex mathematical structure*? This question may be rephrased as: how do mathematicians overcome such difficulties? More precisely, how did they produce such *complex structures*? It seems difficult to answer this question in CF since this theory takes mathematical structures for granted: CF "favors (...) models that accord an essential role to the very mathematical concepts" [Vergnaud, 1990, p. 146]. We could evoke the *affective and dramatic dimension of the situation* but the discussion of Walkerdine [1988] shows that this would be just an attempt to *graft context* into CF. A new approach is needed.

In SF, that is, from the point of view of production of meaning, S1 and S2A are completely different from S2B. Meaning is produced in the *dialectics of the subject and the other* [Lacan, 1973, p. 239]. The motor of this dialectics is a certain demand of the other, before which the subject talks and acts. In S2B the demand required an immediate verbal answer: "mistakes" were promptly available as raw material for meaning production. The social agent in charge of the room hailed with a promised *yes* for an answer. Since he was recognized as an *invited lecturer*, this *yes* was perceived as stemming from a bigger legitimating cultural instance. Therefore desire and fantasy were directly melt with the *power of mathematical discourse*. On the other hand, in S1 and S2A the meaning of a winning strategy would only come in the aftermath, as an *external reference*: a medal. "The focus of their fantasy upon the external reference called up by the practice prohibits the possibility of their driving enjoyment from the power of mathematical discourse" [Walkerdine, 1988, p. 195].[6]

Distinction of the G1A and G1B problems can also be made in SF. We argue that justifying composition of additive operators by resorting to operator states, or by doing away with them, are different *modes of producing meaning*. These justifications belong to different *semantic fields* [Lins, 1994, p. 185]. However it is hard to explain the different degrees of difficulty of G1A and G1B in SF without resorting to some idea of structure.

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[6] This argument also explains why in S2, only upon insistence was G3B solved, and in S1 only in the work-sheet situation was G1A first considered. Non-systematic experiences with G2 indicate that children do not take very long in order to solve G2A by borrowing some money from the bank. Indeed, the demand for a solution of G2A is explicit in the rules of the game.

Briefly, SF indicates the condition that a game situation should satisfy in order to be didactically effective: the implication of the mathematical discourse with the subject's fantasy and object of desire. In order to explain how this condition is present in well succeeded game didactical strategies such as in Giménez [1993], we have to consider how the promised *yes* (the *transfer*, in lacanian terms) is dealt with in a *socially and culturally meaningful context*. The language paradigm is rapidly becoming a central issue in Mathematical Education [Rogerson, 1994]. Further applications of G1-G3 should take such an issue into consideration.

BIBLIOGRAPHY

Baudrillard, J. (1979) *De la Séduction.* Paris. Ed. Galilée.

Brousseau, G. (1983). *Les Obstacles Épistémologiques et les Problèmes en Mathématiques*. Recherches en Didactique des Mathématiques, vol. 4, # 2, p. 165-198.

Dienes, Z. P. (1969A). *Opérateurs additifs.* Paris, O.C.D.L.

______, Z. P. (1969B). *Opérateurs multiplicatifs.* Paris, O.C.D.L.

Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht, Reidel.

_________, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Dordrecht, Reidel.

Giménez, J. (1993) *Aprendiendo Álgebra a través de Juegos.* Barcelona. Universitat Rovira i Virgili.

Glaeser, G. (1981). *Épistémlogie des nombres rélatifs*. Recherches en Didactique des Mathématiques, vol. 2, # 3, pp. 303-346.

Lacan, J. (1973). *Le séminaire de Jacques Lacan. Livre XI: les quatre concepts fondamentaux de la psychanalyse –1964.* Paris: Editions du Seuil.

Lins, R. C. (1994). *Elicting the meanings for algebra produced by students: knowledge, justification and Semantic Fields*. XVIII PME, Lisboa, Vol III, p. 184-191.

Lytle, P. A. (1994). *Investigation of a model based on the neutralization of oposites to teach integer addition and subtraction*. XVIII PME, Lisboa, Vol. III, p. 192-199.

Papy, F. (1968). *Mathématique Moderne.* Bruxelles, M. Didier. vol. 1.

Rogerson, A. (1994). *Symbols as Cultural Communication - A Historical and Didactical Perspective.* 46th International Meeting of the CIEAEM, Toulouse, July 10-16, 1994.

Vergnaud, G. (1990) *La Théorie des Champs Conceptuels*. Recherches en Didactiques des Mathématiques, vol. 190, n. 13, pp. 133-170, 1990.

________ G. (1982) *Cognitive and Developmental Psychology and Research in Mathematics Education: some theoretical and methodological issues*. For the Learning of Mathematics, Vol. 3, #2, November 1982, p. 31-41.

Walkerdine, V. (1988). *The Mastery of Reason.* London, Chapman & Hall, Routledge